

## Eliciting Learner Errors and Misconceptions in Simplifying Rational Algebraic Expressions to Improve Teaching and Learning

Judah P. Makonye<sup>1</sup> and Nzima Stepwell<sup>2</sup>

<sup>1</sup> *Marang Centre for Science and Mathematics Education, University of Witwatersrand, Number, 27 St Andrews Road, Parktown, 2050, Johannesburg, South Africa*  
*Telephone: +27117173206, Fax: +27862076533, E-mail: Judah.makonye@wits.ac.za*

<sup>2</sup> *Northview High School, Athol Road & Northview Street, Highlands North, Johannesburg 2192, South Africa*  
*Telephone: +2783 6118 070, E-mail: stepwell.nzima@yahoo.com*

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**ABSTRACT** This study determined the level to which the learners' errors and misconceptions were reduced when a teaching intervention was directed at their learner challenges. Data was collected from a sample of four boys and six girls of varying mathematical ability from a grade 10 classroom. Participating learners wrote pre-intervention tasks on simplifying rational algebraic expressions. After identifying the errors and misconceptions exhibited by the learners, lesson plans were developed, followed by teaching, which focused on the identified errors and misconceptions. Post-intervention tasks were then written with similar items to the pre-intervention tasks. Post intervention results showed that teaching directed at identified errors and misconceptions helped reduce the errors learners make on average. However, there are some errors that remained difficult to remediate as they kept on appearing even among good mathematics learners. The study recommends further research on algebraic errors, which could not be alleviated by the teaching intervention.

### INTRODUCTION

Algebra is an important component of mathematics and yet, "often the first mathematics subject that requires extensive abstract thinking, a challenging new skill for many students" (Star et al. 2015:1). Despite that it is abstract, it is a very important component of mathematics, which links all the topics of mathematics. As Star et al. (2015) pointed out; algebra is found in all branches of mathematics and science. Elements of algebra are found in all mathematics topics namely, geometry, statistics, trigonometry, vectors, and matrices at basic level, calculus, linear and abstract algebra, topology and differential equations to name a few. It can be confidently concluded that no mathematics of any value beyond arithmetic can be done without the lens of algebra. Therefore, a good understanding and proficiency in algebra can help learners in acquiring other mathematical concepts. A weak learner in algebra struggles to handle mathematics topics. Such a learner could be frustrated and end up giving up on mathematics regarding it as difficult and senseless.

Algebra, which uses abstractions such as letters to stand for numbers that are not known

or to take any value or things that are not known, has shown to be important not only in developing mathematical knowledge, but also useful in real life situations to solve real life problems in business, economics, medicine, science, engineering and all other aspects of real life (Rittle-Johnson 2012; Fuchs 2014). Although people can live without algebra, Usiskin (2004:148) showed the importance of algebra in his analogue that, "...if you visit Mexico but you do not know Spanish, you can get along, but you will never appreciate the richness of the culture and you will never be able to learn as much as you could if you knew Spanish". According to Usiskin (2004), a lower understanding of algebra may lead someone to be unable to understand ideas discussed in the sciences, economics, business, psychology and in real life situations, and this can lead to making unwise decisions.

Elementary or school algebra is mandatory for studying certain courses after school. A lack of understanding of algebra leads to limited career opportunities (Star et al. 2015). As students move to higher learning institutions, acceptance into particular study areas require mathematics, which has a lot of algebraic aspects. Students

who have not studied algebra, or have a weak proficiency in algebra find themselves excluded from completing particular courses. Besides, for future studies and careers, a sound understanding of algebra is essential for employment purposes. Employers prefer to employ creative people who are able to solve problems.

In South Africa, since the introduction of the Annual National Assessments (ANA) examinations, grade 9 learners have persistently performed badly in mathematics. From the grade 9 ANA results, it is found that generally the learners who proceed to do mathematics in grade 10 are weak and need a lot of support in order to improve. In the school that one of the researchers taught, most learners choose to do Mathematical Literacy when they proceed to grade 10. Choosing Mathematical Literacy is limiting in future studies and in the competitive world for employment. Most learners who choose mathematics, the appropriate mathematics, struggle with it and drop it as they proceed to grades 11 and 12. By the time they write Matric in grade 12, only a smaller number remain studying mathematics as compared to mathematical literacy, and the pass rate in mathematics is low. In the Center for Development and Enterprise (CDE), Spaul (2014:5) reports that, "...inspection of school data shows that of the 100 pupils who start grade one, 50 will drop out before grade 12 (most of which happens in grade 10 and 11)..."

During subject meetings, mathematics educators in the department often complain of the lack of understanding by learners of basic mathematical concepts. This is a traditional passive-reception view to learning mathematics, which Sriraman and English (2010) have pointed out is alive among education professionals. If indeed the educators would have told the learners the mathematical concepts, then there is a need to change the way mathematics is taught and select methods that engage learners in activities that would encourage them to develop their own powerful and connected knowledge they can use in developing more powerful knowledge (Hatano 1996). Although learners have the right to subject choices, due to frustration by the learners' performance, educators recommend some learners to take mathematical literacy instead of mathematics. Is this not taking an escapist point of view to learners' errors and misconceptions in which educators label the learners to be dim, to be of low intelligence, and of low mathematical aptitude? Would that not misplace potentially capable mathematics learners

into mathematical literacy classes, because the learners' thinking has not been used to improve teaching and learning?

Because educators in schools are directly in contact with the learners every school day, it is the researchers' belief that if they invest time to determine what their learners do not understand, their errors and misconceptions in algebra, the key to mathematics, they can use that knowledge to create conditions that can enable the learners to reconstruct their thinking, in line with the modern trends of teaching and learning, which encourage the learners to construct their own knowledge (Hatano 1996). How can diagnostic results help learners understand mathematical concepts better, since teaching should build on the knowledge learners already have?

### Objectives

The two main purposes of the research study are to:

- ♦ Identify the errors and misconceptions that Grade 10 learners make when simplifying rational algebraic expressions.
- ♦ Find out to what extent teaching focused on the identified errors and misconceptions helps reduce the identified errors and misconceptions.

### Significance of the Research

The research helps develop teaching competencies hinging on eliciting learners' thinking to developing learner centered teaching approaches, which welcome the errors and misconceptions learners make. This article may inspire readers to appreciate, develop and adopt new ways of teaching mathematics, which encourage the importance of learners constructing their own knowledge. Historically, through errors and misconceptions, "Some of the greatest thinkers-mathematicians and scientists have made mistaken conjectures, many of which have led to new discoveries and broadening their respective fields" (Posamentier and Lehmann 2013:15).

### Literature Review and Theoretical Framework

The constructivist theory of learning (Piaget 1970) informs this research. Learners' errors and misconceptions in learning mathematics evidence of their construction of knowledge since teachers do not explicitly teach them these (Ha-

tano 1996; Nesher 1987). At the heart of constructivism as a learning process is the understanding that knowledge is constructed by the learners individually, as new knowledge interacts with their existing knowledge (Hatano 1996; Makonye and Luneta 2013; Ernest 2010). Unlike the behaviorists' theory of learning, which argues that knowledge can be transferred as a ready-made product to passive recipients, constructivism argues that knowledge cannot be transferred. Instead, knowledge is actively constructed and restructured by each individual learner (Hatano 1996; Makonye and Luneta 2012; Ernest 2010).

According to Ernest (2010:40), knowledge cannot be transmitted. This is one of the first principles of constructivism. Also "Knowledge is not passively received but actively built up by the cognizing subject". When learning mathematics, constructivism proposes that learners must actively engage in the construction of their own knowledge without being told. Such ways of teaching and learning mathematics have proved to be promising ways of developing mathematically proficient learners, according to ideas of realistic mathematics education (Makonye 2014). Learning occurs in the mind of an individual when new experiences or new knowledge comes into contact with the learners' existing knowledge (Makonye and Hantibi 2014; Ernest 2010).

When studying mathematics, learners will interpret new knowledge using their existing knowledge. Ernest (2010:40) indicated that constructivism considers that, "knowing is active, that it is individual and personal and is based on the previously constructed knowledge". In other words, when learning mathematics, new mathematical ideas are interpreted and meaning is made using the existing knowledge from previous learned mathematical ideas, from other subjects or from informal everyday knowledge. This explains the existence of errors and misconceptions learners make when doing mathematics and is in-line with Hatano (1996:207) who pointed out that the procedural bugs and misconceptions learners make are "produced because learners do not swallow given rules or algorithms, but they construct something subjectively". The errors and misconceptions learners commit when doing mathematics have to be welcome in constructivism and be used as a starting point in the teaching and learning process to help lean-

ers construct and restructure their knowledge. Nesher (1987) expressed the importance of tolerating errors and misconceptions in learning and teaching as she suggested that the students' expertise in the learning process is that of making errors.

Errors and misconceptions in mathematics occur when the learners, through assimilation overgeneralize. According to Makonye and Luneta (2013:917), overgeneralization occurs, "When a learner meets a new, mathematical object, he/she might think that the mathematics object belongs to a class, which he/she already has, and so act according to what they already know. The learner assimilates the new mathematical object in the existing class and operates on it as he/she does to the objects in this class". For instance, in low grades, at primary school, addition and subtraction of fractions starts with fractions with a common denominator. When fractions added or subtracted have a common denominator, numerators add or subtract as they have the same name or common denominator. If this is not carefully taught, learners generalize this idea, and so add or subtract numerators and denominators when they come across addition or subtraction of fractions with different denominators, thereby making errors.

On the other hand, accommodation occurs when new knowledge or ideas are different from the existing ideas. When this happens, learners reconstruct and reorganize their existing cognitive schemas to accommodate the new ideas (Makonye 2012). In order for learners to develop mathematical knowledge, cognitive conflict is important in the process of adaptation. According to Makonye and Luneta (2013:117), "it is unlikely that an individual on his or her own will be dissatisfied with his or her own existing knowledge unless influenced by some external forces". Therefore, in the learning process, and in line with Vygotsky's sociocultural theory, it is important that learners engage in discussions with the knowledgeable others.

This statement suggests an important social factor in the learning process suggested by Vygotsky (1978). Learning according to Vygotsky (1978) occurs in the important Zone of Proximal Development, that is, the gap between what learners can do without the assistance of the knowledgeable individuals and the potential development, which the learners arrive at through the help of the knowledgeable others.

The social context of the learning environment, especially one which focuses on the interpersonal relationships between the learner and the teacher is therefore of great importance in the teaching and learning process. Classroom interactions, which allow learning to be negotiated, through collaboration and discussions are of great importance. The discussions learners engage in a social context with the knowledgeable others make it possible for the learners to experience cognitive conflict (Makonye and Luneta 2013). Cognitive conflict occurs when a learner experiences new knowledge, which does not fit reality, or what the learner knows, leading the learner to experience tension in the mind. Piaget (1970) suggested that when learners experience cognitive conflict, they strive to achieve a cognitive balance through the process of equilibration. It is only when learners experience cognitive conflict that the learners may question their beliefs and so begin to search for the truth that brings cognitive balance, a state of equilibration through perturbation (Makonye and Luneta 2013; Ernest 2010). According to constructivism, which considers prior knowledge to be important in the learning process, "...students will draw on previous and concurrent learning from other areas to work with algebraic symbols" (Stacey and MacGregor 1994:290). Therefore, when learners make errors, it does not mean that they are stupid (Nesher 1987).

### ***Effective Mathematics Teaching and the Benefits of Diagnostic Teaching***

The research involved intervention by teaching directed at the errors identified in the pre-intervention tasks. Effective teaching in the mathematics classroom involves the use of constructivism ideas, which regard prior knowledge to be of great significance in the learning process, by giving learners opportunities to construct their own knowledge. Teaching intends to change, reorganize, restructure learners' minds, thereby building more powerful knowledge (Hatano 1986). An emphasis is made on teachers to give learners in mathematics classrooms opportunities to construct their knowledge (Hatano 1996). For effective mathematics teaching, the modern mathematics classroom requires the educators to create conducive environments that give learners space to freely engage in mathematical discussions without fear. In line with

Nesher (1987), learners' errors and misconceptions need to be tolerated and embraced so that they are free to express their ideas as they participate in mathematics community of practice discussions, where the mathematics is the authority instead of the educators (Ross 1998).

### ***Types of Errors Learners Make When Simplifying Rational Algebraic Expressions***

In simplification of algebraic expressions, errors learners make are related to their prior knowledge on common fractions and also are strongly caused by learners depending more on cues rather than understanding (Figuera et al. 2008). This shows the importance of the two different understandings that are related to learning identified by Skemp (1976) (Relational understanding and instrumental understanding). According to Figuera et al. (2008), the reliance on cues when simplifying rational algebraic expressions is in line with the use of Skemp's instrumental understanding, which requires learners to follow certain procedures without understanding as they simplify algebraic expressions.

As the learners do so, they retrieve wrong, incomplete, inappropriate or flawed rules that lead them to make errors. In their research on errors learners commit when simplifying rational algebraic expressions, Figuera et al. (2008) and Mhakure et al. (2014), identified several errors and misconceptions learners make due to their prior knowledge and the dependence on instrumental understanding, which requires procedural understanding, and prior knowledge on simplifying common fractions. Such errors included the cancellation error, partial cancellation, like term error, linearization and de-fractionalization and equationalization.

#### ***Cancellation Error***

The cancellation error according to Figuera et al. (2008) occurs when there are like terms or expressions in the numerator and the denominator. Learners tend to cancel these as if the like terms or like expressions were common factors based on their prior knowledge when simplifying common fractions. For instance, when simplifying a rational expression  $\frac{4x^2-1}{4x^2+2x}$  learners perceive the  $4x^2$ , the like term in the numerator and the denominator as a common factor and use to cancel as shown below based on their previous knowledge when working with rational numbers.

### *Partial Cancellation Error*

According to Figuieras et al. (2008), this error is characterized by division only taking place between some terms in the algebraic expression. For example,  $\frac{4x+12}{4}$  is simplified to obtain  $3x+12$  as learners divide  $4x$  by  $4$  and also  $4$  by  $4$ . The expression could also simplify incorrectly to obtain  $4x+3$  if  $12$  in the numerator is divided by  $4$  and  $4$  in the denominator is divided by  $4$ .

### *De-fractionalization*

The error is characterized by learners' transformation of fractions that have unitary numerators to non-fractions (Figuieras et al. 2008; Mhakure et al. 2014). For instance,  $\frac{1}{x}$  is simplified to obtain  $x$ .

### *Linearization*

The linearization error according to Figuieras et al. (2008) involves learners breaking up rational expressions with compound denominators into separate fractions. However, the fractions are broken up incorrectly. For example, is  $\frac{4}{4x+8}$  broken up incorrectly to give two fractions,  $\frac{4}{4x+8} + \frac{4}{8}$ .

### *Like Term Error*

The like term error, according to Figuieras et al. (2008) and Mhakure et al. (2014) involve learners performing the subtraction operation instead of the division operation. For instance,  $\frac{2x}{8}$  simplifying to obtain  $2x-8x = -2x$ .

### *Equationalization*

According to Mhakura et al. (2008), equationalization involves learners transforming rational fractions into equations which they want to solve.

## **MATERIAL AND METHODS**

According to Cohen and Manion (1994), the research design involves the researcher's plans on how to go about understanding a phenomenon. It involves the whole research process, the

research approaches, procedures, data collection, and sampling methods used as the researcher attempts to find solutions to the research questions. Ethical issues are also of great importance in order to carry out a research.

The qualitative research as opposed to the quantitative research considers those life phenomenon are just too complicated to be explained by positivist research only. Qualitative research is descriptive and seeks for explanations in understanding a phenomenon or reality, rather than the causal quality of the quantitative research. Qualitative research therefore accepts and recognizes individual interpretations on reality based on the responses and observations obtained from participants (Cohen and Manion 1994). Qualitative research regards people as living and working in social groups with different beliefs, cultures, traditions and ways of life. Through interactions with the people, such as talking to people, listening to what they say and observing the way they do things, researchers can learn a lot from the observed participants about questions of interest (Cohen and Manion 1994). The world is made up of people with their own assumptions, beliefs and values and that the way of knowing reality is by exploring the experiences of other people regarding a specific phenomenon. Through, subjective explanations of the observed participants, researchers can learn a lot about the participants. Qualitative research techniques require that the researchers and the participants interact. Instead of seeing the world as containing hard tangible realities with theories that already exist, which only need to be approved or refuted by scientific and objective means, qualitative research gives room to the development of such theories as researchers engage naturally with the participants.

## **Methods and Techniques**

In line with the aims of the research, the pre-intervention written tasks on simplification of algebraic expressions were used to identify the errors and misconceptions learners make when simplifying algebraic expressions. In the post-intervention tasks the researchers aimed to find to what extent errors and misconceptions were reduced after teaching that was directed at the identified errors and misconceptions in the pre-intervention task.

In the intervention, one researcher asked questions which required learners to discuss and construct their own mathematical knowledge on simplification of rational algebraic expressions. Learners' contributions were carefully noted as well how they interacted to find out the errors they made. Learners were helped to correct their misconceptions.

### Sampling

According to Maree (2013:79), "sampling refers to the process used to select a portion of the population for study". There are basically two types of sampling methods used in research (Maree 2013). These are probability sampling methods and non-probability methods. In probability sampling, all members of the population have an equally likely chance to be selected (Maree 2013). In non-probability methods, selection is not representative; rather sampling is purposive and convenient. In qualitative research, sampling is generally non-probability. The researchers chose the sample with the belief that it would give the data that is required in the study (Cohen and Manion 1994; Maree 2013). In purposive sampling, sampling is done with a specific purpose in mind while convenience sampling refers to situations in sampling where the population selected is based on easy and conveniently availability of the participants (Maree 2013). Convenience sampling was also used in the selection of the participants as participants from two grade 10 classes at the school where one researcher taught for purposes of time and reduction on costs.

### Tasks

The selected items for the tasks were meant to find out the kind of errors learners make when simplifying rational algebraic expressions (See Appendix A: Tasks A to Q). For instance, item 2.2 (D)  $\frac{5x+15}{5}$  was asked to find out if learners could perform incomplete cancellation and also conjoining. Validity and reliability of tasks was considered to be important. The items in the tasks had to be in-line with the grade 10 work schedules and curriculum statement. This was to ensure learners are given tasks that are to their level. The learners were expected to be able to work out the kinds of items in the tasks. To en-

sure validity and reliability, the tasks were also moderated by a colleague in the school where the researcher works. Memorandums were made for the tasks in order to check on the possible correct responses. This was to guide the researcher to notice the errors and misconceptions learners made when simplifying rational algebraic expressions.

### The Intervention

The intervention activities were divided into three break time lessons of 30 minutes each. Various strategies to help learners overcome and reduce the errors they had committed in the pre-intervention tasks were implemented. The researchers were guided by Vygotsky's sociocultural theory, which considers that talk is important in the learning process, and constructivism, and that learners have to construct their own knowledge. Learners engaged in whole group discussions and pairs as they played a bingo game. In order to encourage learners to freely engage in discussions where they would express their ideas without fear, their pre-intervention scripts had no marks indicating where they had made errors or where they were right. Any learner was requested to give an explanation on how they simplified an item from the pre-intervention tasks. Their presentations were followed by whole group discussions of the solutions. This gave the learners an opportunity to discover their own errors and misconceptions. After whole group discussions, learners then worked in pairs to simplify expressions on pair worksheets. Pairs helped each other to simply algebraic expressions on a card selected. A pair would shout bingo if content with their solution and so took the opportunity to explain their solution to the group for points. If they were errors and misconceptions in the solutions, it sparked whole group discussions and the pair lost points. This encouraged learners to work together in overcoming errors and misconceptions.

## RESULTS

### Analysis of Errors and Misconceptions Learners Displayed in the Pre-intervention Tasks

After writing the pre-intervention tasks, each participant's script was analyzed to identify the errors they made. The analysis noted that par-

Participants had misconceptions when simplifying rational algebraic expressions, which made them incorrectly simplify algebraic expressions. Table 1 shows the results in percentage of the learners' performance individual pre-intervention items A to Q. An item was considered to be incorrect when simplified incorrectly, when simplified incompletely or when no response was given. Although the errors and misconceptions could not be identified when there were no responses given to some items in the tasks, the assumption was that learners could not respond because they had misconceptions. Table 1 shows that on average, more than eighty-five percent of the learners failed the pre-test and that they held many misconceptions on the algebraic tasks written.

In the following paragraphs, the researchers present the types of errors that were made by learners when simplifying rational algebraic expressions using the typological framework by Figueras et al. (2008).

### **Conjoining Errors**

One common error that was made by the participants was conjoining. Conjoining is the senseless combination of unlike terms by multiplication when adding or subtracting algebraic expressions. This error was exhibited by learners of various learning abilities among the par-

ticipants in the pre-intervention tasks, and during the intervention activities.

### **Incorrect Cancelling**

Another error that was common among the participants was that of incorrect cancelling. This error like conjoining was committed by all the different ability participants, including the good learners in mathematics.

### **Incomplete Cancelling**

Another common error that was committed by learners of various capability levels, including the good learners and is similar to incorrect cancelling was that of partial cancelling. Below are some examples of incorrect canceling errors.

### **De-fractionalization**

Another common error participants committed during the simplification of rational algebraic expressions was the de-fractionalization error. Figueras et al. (2008) identified it as the error where learners remove the numerator of a fraction if there happens to be a numerator.

### **Like Term Error**

Another error that was made by the participants was the like term error (Figueras et al.

**Table 1: Learners' performance in the pre-intervention task**

<i>Item</i>	<i>Number of learners who got item correct</i>	<i>% of learners who got item correct</i>	<i>Number of learners who got item incorrect</i>	<i>% of learners who got item incorrect</i>
A	3	30	7	70
B	4	40	6	60
C	3	30	7	70
D	5	50	5	50
E	3	30	7	70
F	1	10	9	90
G	2	20	8	80
H	1	10	9	90
I	2	20	8	80
J	5	50	5	50
K	0	0	10	100
L	0	0	10	100
M	3	30	7	70
N	3	30	7	70
O	0	0	10	100
P	3	30	7	70
Q	0	0	10	100

Source: Authors

(2008). Learners who commit the like term errors perform a subtraction instead of division operation.

### ***Addition or Subtraction of the Numerators***

Various errors were made by learners when simplifying ordinary common fractions (items 1.1 and 1.2) as shown below. Learners applied this prior knowledge inappropriately when adding or subtracting fractions.

### **Some Discussions with Learners**

Researcher: *So can you explain how you worked here?*

(That is Item C,  $a + \frac{a}{3} = 3.a + \frac{a}{3}$ ,  $3 = 3a+a=4a$ )

Learner P: *What I did sir, I found the LCD. So my LCD is equal to 3 and then I multiplied my LCD in both sides. And then I said a times three+a over three times three. I want to get rid of my denominator. Then equal to 3a plus a. Then two variables, they are like terms and so my answer is four a.*

Researcher: *Are you convinced with the explanation?*

(Some learners nod their heads to show they were convinced by Learner T explanation though incorrect).

Learner M: *Eihe* (In agreement that she is convinced)

Learner M: *May be.*

Teacher: *Why?*

Learner M: *I got 4 a over 3* (Volunteers to show her working)

(Learner M asks a question)

Learner M: *Sir, the thing is regarding this algebraic expression, usually when you want to solve these expressions you have to find the LCD... There I do not know how, why was the denominator removed in the first place?*

Her presentation surprised many learners and they thought their solutions were incorrect as it was convincing. In terms of constructivism, the learner held the beliefs she explained to the class as affected by prior knowledge. The beliefs could be true or false. Through interactions and discussions with other learners, her schema were restructured and reorganized to develop more powerful knowledge. This occurred as one Learner M asked whether there was a need to remove the fractions.

Another strategy that was used to help learners to overcome their errors was the use of real numbers. For instance,  $\frac{a}{3}$  was replaced with 1 and  $1 + \frac{1}{3}$  in order to find out whether the removal of fractions helped simplify algebraic expressions. This was perhaps used to create cognitive conflict necessary in learning that follows the ideas of constructivism. Cognitive conflict occurs when what is obtained is not the same with the right thing or what is intended to be obtained. Closely linked to the use of real numbers was the substitution with real numbers for the variables in the original expressions and the final simplified expressions to check if the simplified expression was right. This strategy was suggested by the participants themselves as they engaged in mathematical discussions.

Collaborative learning, in line with constructivism is of great importance in the learning process, as learners learn from each other by engaging in discussions with the knowledgeable others. The researchers placed learners in mixed ability groups of three learners each to simplify the expressions that were on group activity cards. An emphasis was made that each group member had to take a lead in simplifying the rational algebraic expressions on the group activity cards. The group activities revealed that some errors the learners had committed in the pre-intervention tasks continued to be committed. For instance, learners continued to incorrectly cancel.

During the intervention activities, it was discovered that learners committed some errors they had committed during the pre-intervention at some stages of simplifying some algebraic expressions. For instance, when simplifying the rational expression  $\frac{4x^2-1}{4x^2+2x}$  learners were not attracted to the like term  $4x^2$  in the numerator and the denominator at first. They factorized correctly to obtain

$$\begin{aligned} & \frac{(2x+1)(2x-1)}{2x(2x+1)} \\ &= \frac{2x-1}{2x} \\ &= \frac{-1}{2x} \end{aligned}$$

They incorrectly cancelled in the second step where they were attracted to  $2x$  in the numerator and denominator, and perceived it as a common factor. The incorrect cancelling error reappeared at different stages of the simplifying process. Another error that kept on occurring during the intervention activities was that of incomplete cancelling, where learners were required to factorize.



An interview was conducted with Learner N, one of the learners who committed the error of de-fractionalization to find out why the learner de-fractionalized in item 2.7 to find out how she obtained  $x-1$  instead of  $\frac{1}{x-1}$  as shown below.

#### *De-fractionalization Error Discussion*

Researcher: *Would you like to explain to me how you simplified item 2.7, that is,  $(\frac{x+1}{x^2-1})$ .*

Learner N: *Ok sir, I saw  $x^2-1$  and I started factorization, which I now feel it was a mistake.*

Researcher: *What was the mistake?*

Learner N: *Is it a positive or a negative?*  
(This is because the sign in the denominator was cancelled and so was not clear.)

Researcher: *Negative?*

Learner N: *I wanted it to be positive so that's why I factorized.*

Researcher: *Suppose it is negative, because it was negative. Let's keep it negative.*

Learner: *Then I factorized. It would be  $x$  plus one,  $x$  minus one. I saw that  $x$  plus one and  $x$  plus one are the same, like terms then I cancelled and got one.*

Researcher: *When you cancel like terms what happens?*

Learner N: *The answer remains  $x-1$ .*

Researcher: *What does it mean to cancel like terms?*

Learner N: *Sir, I think that if you have like terms they should be taken away. Ok sir, it's like  $x$  plus one divided by  $x$  plus one its one. One times  $x$  minus one is  $x$  minus one.*

The interview revealed that after factorizing the denominator using the difference of two squares, she redefined the expression,  $\frac{x+1}{(x+1)(x-1)}$  to be  $\frac{x+1}{x+1} \times x-1$  causing the learner to make an error.

Another misconception that was revealed in the interview is that the learner considered the expressions in the numerator and the denominator as like terms and so used to cancel instead of finding the common factor. Also, the interview revealed that there is a possibility that learners may understand cancelling according to everyday knowledge as the learner pointed out that to cancel meant, "Like terms should be taken away". Such ways of understanding cancelling could cause errors when simplifying rational algebraic expressions.

#### **Analysis of the Post-intervention Task**

After writing the post-intervention tasks, the learners' scripts were analyzed to find out whether teaching that focused on the identified errors in the pre-intervention tasks helped learners reduce the errors they committed.

Table 2 shows the learners' performance in the post-intervention tasks by item. The analysis in Table 2 shows an average error reduction rate of 35.8 percent in the post-test. This shows a significant reduction in errors and therefore misconceptions, which researchers attribute to the teaching intervention targeting learners' errors and misconception noted in the pre-test.

**Table 2: Results table post-intervention tasks**

Item	% errors pre-test	% errors post-test	% Reduction
A	70	50	20
B	60	50	10
C	70	40	30
D	50	10	40
E	70	20	50
F	90	20	70
G	80	30	50
H	90	20	70
I	80	40	40
J	50	20	30
K	100	80	20
L	100	90	10
M	70	20	50
N	70	10	60
O	100	80	20
P	70	50	20
Q	100	80	20

Source: Authors

## **DISCUSSION**

In the following paragraphs, the researchers discuss the results of the pre-intervention tasks, post intervention tasks and the intervention activities using constructivism, Vygotsky's socio-cultural theory and Skemp's instrumental and relational understanding ideas.

### **Conjoining**

As indicated, one error learners committed in the pre-intervention task was conjoining. With regard to constructivism and in line with Star et al. (2015), learners who conjoined could not ac-

cept to leave the answers as algebraic expressions. According to Star et al. (2015), Rittle-Johnson (2012) and Fuchs (2014), and in line with constructivism, which regards the learners' prior knowledge to be of significant importance in the learning process, the learners' previous arithmetic knowledge, where operations on numbers yield one numeric answer influenced and caused the learners to conjoin. In particular, Fuchs (2014) argued that students' competency in arithmetic does not transfer to competency in algebra.

Another cause for conjoining by the learners is the lack of understanding of algebraic expressions as objects, which could be considered as processes or as objects (Rittle-Johnson 2012). Linked to their prior arithmetic knowledge, the learners tend to perceive algebraic expressions only as processes and fail to accept the expressions as objects. As a process, from arithmetic knowledge, the operation signs (addition, subtraction, multiplication and division) require an action and the equal sign requires a numeric answer to be provided. The learners perceive the algebraic expression as processes and not as objects, thereby leading them to conjoin. According to Makonye and Luneta (2013), and in-line with constructivism, the learners who conjoined managed to retrieve a correct schema from their arithmetic knowledge but applied the correct knowledge inappropriately on algebraic expressions.

### **Incorrect Cancelling**

Learners who cancelled incorrectly were attracted to the like terms or expressions in the numerator and the denominator of the rational algebraic expressions, and perceived these as common factors. This behavior was in line with Star et al. (2015) who pointed out that *like terms* in simplification of rational algebraic expressions provide learners with attractive visual cues, which cause them to cancel incorrectly. For instance, in item E ( $\frac{4x-20}{12+4x}$ ), Learner G was attracted to  $4x$  in the numerator and the denominator and treated the common term in the numerator and the denominator as a common factor. This also occurred in item G where  $6x$  in the numerator and in the denominator was considered to be a common factor. In terms of constructivism, the learner's arithmetic prior knowledge made them to error. In simplifying arithmetic common frac-

tions, learners use the common factor to cancel, and the learner seems to be able to do that.

### **Partial Cancelling or Incomplete Cancelling**

The learners who committed the error had the correct schema that division of fractions involves simplification of the fraction by dividing the numerator and the denominator by a common factor. The learners retrieved the right schema to simplify the expression from their prior knowledge, which is regarded to be important in the learning process by constructivism. However, the learners divided partially leading them to make a partial division error.

### **De-fractionalization**

With regard to constructivism, some learners did not want to work with fractions and so drew on prior knowledge that is used when solving equations involving fractions, to remove the fractions, but did it incorrectly as describe before for Item C.

### **Like Term Error**

In line with constructivism, learners who committed the like term error drew upon prior knowledge based on the laws of exponents. During intervention activities, one learner explained that she subtracted because she used knowledge obtained from the laws of exponents. *"I used all the knowledge I got from the laws of exponents,"* said one learner.

Learner N said, *"I saw that  $7x$  and  $x$  are like terms, so if they are like terms, you add then as there is an addition sign. I got 8...then the law of exponent says that when there is a division sign you minus."* This is in line with constructivism, which regards prior knowledge to be important in the learning process. However, in line with Oliver (1989), the prior knowledge that was retrieved was flawed or faulty, hence leading the learner to commit an error. According to the learner she stated the flawed law of exponent she used as, *"When an equation is in a division for you subtract"*. As Nesher (1987) argued, the learner committed an error, not because she was stupid, but rather she used flawed or faulty prior knowledge.

### Expansion of Already Factorized Expressions

The pre-intervention tasks also showed that learners making unnecessary expansions to algebraic expressions when simplifying already factorized algebraic expressions

$$(Item J: \frac{(x+1)(x-2)}{x+1})$$

In terms of constructivism, the learner expanded and removed the brackets correctly by multiplying. This is based on the prior knowledge, when multiplying algebraic expressions and also when solving equations that involve brackets. However, because the expressions were already factorized, there was no need to expand. The expansion, though correct, was unnecessary and led some learners to eventually make an error as it produced attractive like terms in the numerator and the denominator, which were perceived as a common factor and so were incorrectly used to cancel. According to Skemp (1976), learners unnecessarily expanded without reasoning because of instrumental understanding, which emphasizes executing correct procedures with little understanding and reasoning. In mathematics, learners need to reason for any procedure that they perform.

The post-intervention tasks showed that the errors learners committed in the pre-intervention tasks continued to be committed even after intervention. However, the intervention activities reduced the errors committed by the participating learners, as there was an increase in the percentage performance for most of the items when compared to the performance before intervention. Some errors were difficult to reduce. For example, items K, L, O and Q showed little improvement by learners. All learners did not manage to simplify item Q correctly. According to Makonye and Khanyile (2015), there are some algebraic errors that are very difficult to remediate.

### CONCLUSION

Firstly, the research showed that when grade 10 learners simplify rational algebraic expressions, they commit many errors and show various misconceptions. The various errors learners commit were generally linked to their prior knowledge on simplification of common fractions and other ideas in mathematics such as solving equations and simplifying exponents. In the interview, one learner clearly showed how much prior knowledge influence learning as she point-

ed out that, *"I used all the knowledge I got from exponents"*. In line with research literature, the researchers observed that when learners simplify rational algebraic expressions, using their prior knowledge, they understand that cancelling should be done to the numerator and the denominator using something common. However, the error they make is that the something they use to cancel is not a common factor. They tend to use the like term or similar expressions seen in the numerator and the denominator to cancel. In fact the intervention tasks showed that the learners used the like terms to cancel as many explained, *"these are like terms and so they cancel"*.

The errors learners commit when simplifying rational algebraic expressions included incorrect cancelling, partial cancelling or incomplete cancelling, de-fractionalization, like term error, and conjoining. Incomplete cancelling is another error learners commit when simplifying rational algebraic expressions. From prior knowledge, they knew that they had to cancel with a common factor, they identified the common factor, but they cancelled incompletely. Other error learners committed when simplifying rational algebraic expressions revealed by the research is de-fractionalized. De-fractionalization occurred when learners deliberately removed the denominator as what happens when they have to solve equations involving fractions.

The intervention did not eradicate the errors the participants committed in the pre-intervention tasks. When teaching is directed at errors learners commit, it is important that educators consider that the aim of teaching is not to eradicate or uproot the errors. The errors learners commit are due to their deep beliefs about some mathematical ideas, which are influenced by prior knowledge. These beliefs cannot easily be eradicated, as they are ideas that are held to be true and make sense to the learners. However, intervention is a necessary activity to help reduce the errors as learners restructure reorganize their knowledge that way. Although the errors were not eradicated, the errors learners committed were reduced as shown by the increase in percentages of the learners who got the items correct as compared to the pre-intervention results.

### RECOMMENDATIONS FOR FURTHER RESEARCH

Future research must be focused on algebraic fraction simplification errors that have been

shown to be resistant to interventions in this article; for example cancellation error. This will go a long way to shed light on some key constructs that continue to make learning algebra incomprehensible to many learners.

### LIMITATIONS OF THE STUDY

One of the limitations of the study is that data collection and the teaching intervention was done over a short period of time, that is, in less than two calendar months. The researchers think that a more lengthy longitudinal study would have been more appropriate in that it would have determined whether the gains in learning were lasting. Another assumption could be that students learnt the algebra more because of the intervention. Yet, there could be the researcher effect in that since the researchers were quite prepared, observant and careful in their research, this resulted in learners learning more than usual. This in fact is not a bad thing as it implies that if teachers are conscious and well prepared for their lessons, then their students learn more. It may be important to have a control group to see what happens when a normal re-teaching of the topic was done. This would have enabled a comparison as to whether the teaching intervention focused on learners' errors really was more helpful in reducing learner errors than normal remediation.

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### APPENDIX: Tasks

#### Question 1

1 Workout the following and write the answers in simplified form.

$$1.1 \frac{3}{5} - \frac{1}{2} + \frac{2}{3}$$

$$1.2 \frac{5}{7} - \frac{1}{3}$$

#### Question 2

2. Simplify the following algebraic expressions as far as possible.

$$1.1 \ x + \frac{x}{5} \text{ C} \quad 2.2 \ \frac{5x+15}{5} \text{ D} \quad 2.3 \ \frac{4x-20}{12+4x} \text{ E}$$

$$2.4 \ \frac{15-3c}{3c} \text{ F} \quad 2.5 \ \frac{x(x-6)}{6x} \text{ G} \quad 2.6 \ \frac{7x+x}{7x+3x} \text{ H}$$

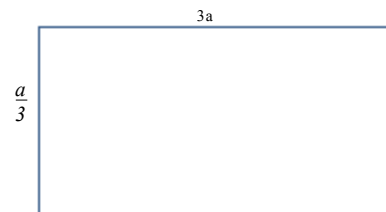
$$2.7 \ \frac{x^2-1}{x-1} \text{ I} \quad 2.8 \ \frac{(x+1)(x-2)}{x+1} \text{ J} \quad 2.9 \ \frac{(b+2) - 3a(b+2)}{b+2} \text{ L}$$

$$2.10 \ \frac{2(a+1)^2 + (a+1)}{a+1} \text{ L} \quad 2.11 \ \frac{a(3-a)}{a} \text{ M} \quad 2.12 \ \frac{5x+5y}{x+y} \text{ N}$$

#### Question 3

3.1 Mother has  $2(a+1)2 + (a+1)$  rands she wants to share among  $a+1$  children. Calculate in terms of  $a$  and in its simplified form the amount each child will get. O

3.2 The dimensions of a rectangle are given as shown in the diagram below. Find the area of the rectangle in terms of  $a$  in its simplified form. P



3.3 Find the perimeter of the rectangle above. Q

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